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Entropy fluctuations for directed polymers in 2+1 dimensions

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We find numerically that the sample to sample fluctuation of the entropy ΔS is a more sensitive tool in distinguishing low from high temperature behaviors than the common corresponding fluctuation in the free energy. In $1+1$ dimensions we find a single phase for all temperatures, since $(\Delta S)^2$ is always extensive. In $2+1$ dimensions we find a behavior that at first sight might appear to be a transition from a low temperature phase where $(\Delta S)^2$ is extensive to a high temperature phase where it is subextensive. This is observed in spite of the relatively large system we use. The observed behavior is explained not as a phase transition but as a strong crossover behavior. We use an analytical argument to obtain $(\Delta S)^2$ for high temperature, and find that while it is always extensive it is also extremly small, and that the leading extensive part decays very quickly to zero as the temperature is increased.

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The problem of directed polymers has attracted much interest in recent years. It is relevant to many fields ranging from surface growth phenomena and spin glasses to flux lines in high-T_c superconductors [1–4]. It is well known that the problem of directed polymers in a random medium is equivalent to the Kardar-Parisi-Zhang (KPZ) equation that describes surface growth [5–7]. Much is known about the KPZ system, in particular in $1+1$ dimensions. The KPZ equation provides an exact dynamical exponent for directed polymers in $1+1$ dimensions [5–9]. The situation for higher dimensions is more complex. Traditional approximation schemes like dynamical renormalization-group methods fail to produce the exponents obtained by simulations [1,2,10–15]. A self-consistent expansion of the correlation function introduced later [16] yields results compatible with simulations for $2+1$ dimensions. Above $2+1$ dimensions, the behavior of directed polymers has a phase transition when temperature is raised (in the directed polymer problem or the level of noise in the KPZ system), and the system goes over from a strong coupling behavior to a perturbative weak coupling behavior [17–20]. All the field theoretical treatments agree that at $2+1$ dimensions, namely, at the lower critical dimension itself, no transition should occur, and strong coupling behavior exists at zero and any finite temperature [16,19–21]. Some authors claimed in the past, on the basis of numerical simulations, to have obtained a phase transition in $(2+1)$-dimensional systems [22–28]. The systems studied were, however, relatively small, and all those claims were not pursued eventually. In this paper, we present numerical results obtained for much longer systems in $1+1$ and $2+1$ dimensions. We study the free energy fluctuations that are usually studied in literature, but in addition we numerically obtain the sample to sample fluctuations of the entropy. This is a quantity that is less common in the literature, and was introduced first, to the best of our knowledge, by Fisher and Huse [29]. The fluctuations in the entropy are more pronounced than those of the free energy and, in fact, we expect it to be extensive in the size of the polymer in $1+1$ and $2+1$ dimensions. In higher dimensions we expect a transition from a low temperature extensive phase to a high temperature subextensive one. That is, defining the quantity, $\lambda(t)=(\Delta S)^2/t$, its infinite volume limits, behaves like an order parameter. It is zero at high temperatures and is of order 1 at low temperatures. The numerical results concerning $\lambda(t)$ in $1+1$ dimensions are those that are expected. We find that λ is of order 1 over the entire temperature region. In $2+1$ dimensions, the numerical results seem to indicate an unexpected transition from a low temperature phase in which λ is of order 1 into a high temperature regime where it is approaching zero. Although the transition temperature seems to be defined very sharply, and although $\lambda(t)$ can be fitted at high temperatures by $\lambda(t)\propto \ln t/t$, we claim that it is just a crossover phenomenon, and the reasons for it occurring in such a spectacular way will be discussed later.

Consider a directed polymer on a hyperpyramid lattice structure with the random energy assigned on each bond. The partition function $G(R,t)$ for directed polymers starting from (0,0) and ending at (R,t) is defined by $G(R,t)=\Sigma_C e^{-E_C/T}$, where E_C is the sum of the energy on the path C, and T is the temperature. For simplicity, we demonstrate our calculations using the transfer-martrix method for the case of $1+1$ dimensions. A similar formalism has been used for $2+1$ dimensions. The iteration relation for the partition function $G(R,t)$ is

$$G(R,t+1)=G(R-1,t)e^{-\epsilon_l/T}+G(R+1,t)e^{-\epsilon_r/T}, \quad (1)$$

in which ϵ_l and ϵ_r are the energy assigned to the left and right bonds of the point (R,t). The free energy $F(t)$ is given by $F(t)=-T\ln G(t)$, where $G(t)=\Sigma_R G(R,t)$ is the total partition function. The free energy fluctuation $\Delta F=(\overline{F^2}-\overline{F}^2)^{1/2}$ was commonly studied ($\overline{A}$ is the ensemble average of the quantity A). We can also define the internal energy, $\langle E\rangle\equiv\Sigma_R\Sigma_C E_C e^{-E_C/T}/\Sigma_R\Sigma_C e^{-E_C/T}$. The internal energy

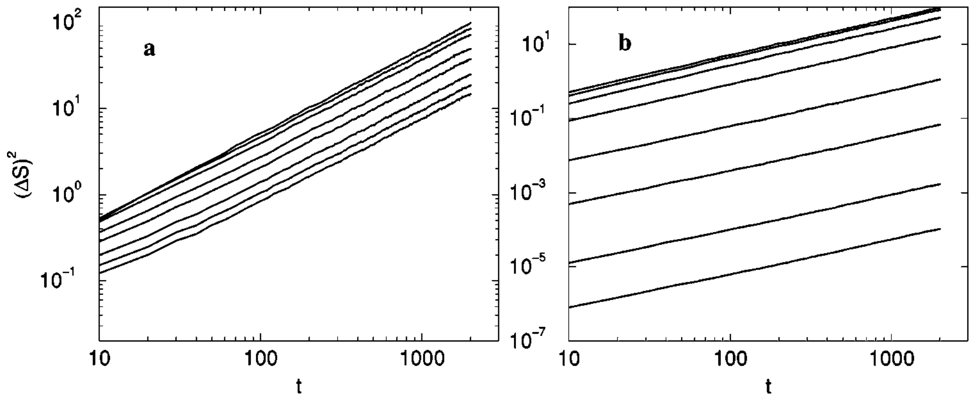

FIG. 1. Plot of the entropy fluctuation as a function of time t for different temperatures in $d=1+1$. (a) For $T=\frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$, and $\frac{1}{50}$ (from top to bottom). (b) For $T=10, 5, 2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$ (from bottom to top).

fluctuation $(\Delta E)_T=(\overline{\langle E\rangle^2}-\overline{\langle E\rangle}^2)^{1/2}$ is also an interesting quantity. In order to obtain the iteration relation for the internal energy $\langle E\rangle$, we define $\hat{E}(R,t)\equiv\Sigma_{C(R,t)}E_{C(R,t)}e^{-E_{C(R,t)}/T}/G(t)$. It is clear that $\langle E\rangle=\Sigma_R\hat{E}(R,t)$. The iteration relations for $\hat{E}(R,t)$ are

$$\hat{E}(R,t+1)=[e^{-\epsilon_l/T}\hat{E}(R-1,t)G(t)$$
$$+e^{-\epsilon_r/T}\hat{E}(R+1,t),$$

$$G(t)+\epsilon_l e^{-\epsilon_l/T}G(R-1,t)+\epsilon_r e^{-\epsilon_r/T}G(R+1,t)]/G(t+1). \tag{2}$$

To understand the difference between the free energy and internal energy fluctuations, we use the entropy $S=(\langle E\rangle-F)/T$. We present systematic numerical simulations for the entropy fluctuation $\Delta S=(\overline{S^2}-\overline{S}^2)^{1/2}$, based upon Eqs. (1) and (2), with the initial conditions $G(R,0)=\delta_{R,0}$ and $\hat{E}(R,0)=0$. The random energy assigned on the bond is assumed to be uniformly distributed in the interval $(-0.5, 0.5)$ and to be uncorrelated in space and time. We use a length of up to $t=2000$ $(d=1+1)$ and $t=1000$ $(d=2+1)$ [30]. Six thousand configurations for $1+1$ dimensions, and 4000 configurations for $2+1$ dimensions, were collected to take the ensemble average.

In $1+1$ dimensions, the numerical results clearly show that the entropy fluctuation has the behavior $(\Delta S)^2\propto t$ for any temperature (see Fig. 1) [30]. It is expected that the entropy fluctuation will tend to zero at limits of zero and infinite temperature. Indeed, the slope of $(\Delta S)^2/t$ [Fig. 3(a)] is about 1 for $T<T_P$, i.e., $\lambda(t)\sim T$, and about -4 for $T>T_P$, i.e., $\lambda(t)\sim T^{-4}$. As a result, the free energy fluctuation and internal energy fluctuation will be the same at the two limits. About $T=T_P=0.2$, the entropy fluctuation reaches a maximum. There is no evidence of a phase transition in $1+1$ dimensions.

The picture of directed polymers in $2+1$ dimensions is more complicated than that of one in $1+1$ dimensions (see Fig. 2). Similar to $1+1$ dimensions, the entropy fluctuation tends to zero at the two limits of zero and infinite temperature, and there is a peak at $T_P\cong 0.11$. For low temperatures $T\leqslant T_P\cong 0.11$, we see that the entropy fluctuation $(\Delta S)^2$ tends to t and $(\Delta S)^2\sim T$ as in $1+1$ dimensions [see Figs. 2(a) and 3(b)]. However, for $T>T_P$, we find that the increase of entropy fluctuation as a function of t becomes slower and slower as the temperature is increased [see Fig.

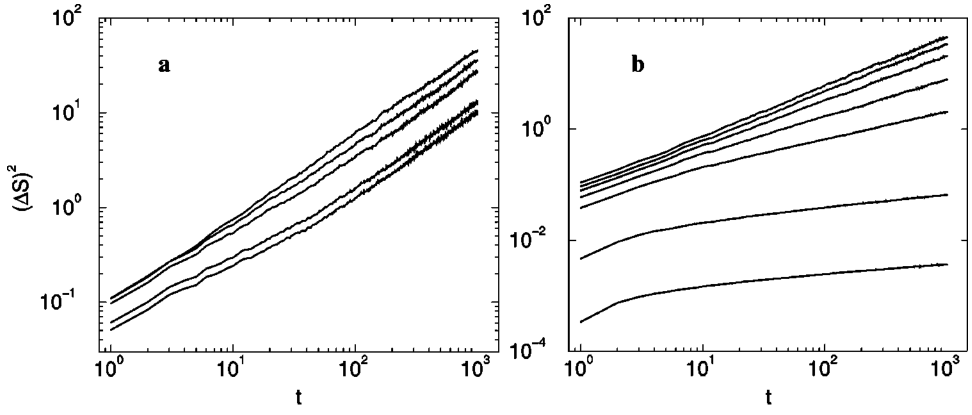

FIG. 2. Plot of the entropy fluctuation as a function of time t for different temperatures in $d=2+1$. (a) For $T=\frac{1}{9.3}, \frac{1}{15}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{50}$ (from top to bottom). (b) For $T=1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$, and $\frac{1}{9.3}$ (from bottom to top).

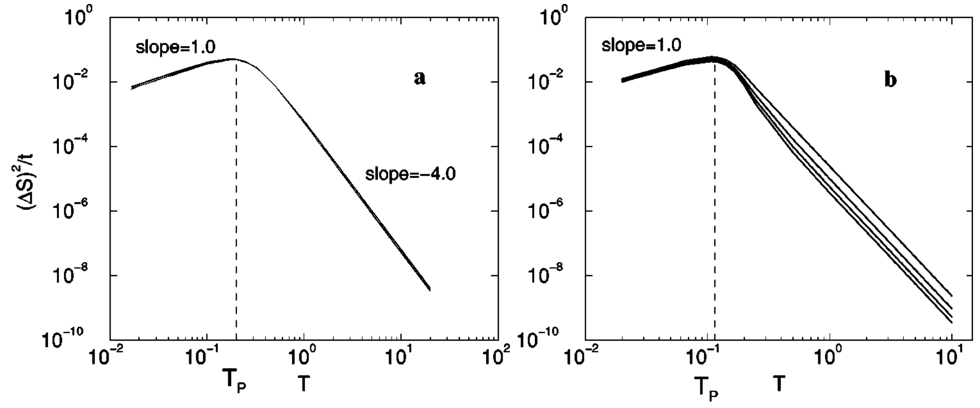

FIG. 3. Plot of the entropy fluctuation per unit length of the polymer as a function of temperature (a) for different times $t = 50$, 500, 1000, and 2000 in $1+1$ dimensions; and (b) for different times $t=100$, 300, 600, and 1000 (from top to bottom) in $2+1$ dimensions.

2(b)]. At very high temperatures, e.g., $T=10.0$, the entropy fluctuation is proportional to $\ln t$ for large t. It seems that $(\Delta S)^2/t$ will tend to the nonzero values only in the region $0 < T \leqslant T_P$. The fluctuations of the free energy ΔF are correlated with the fluctuations of the entropy ΔS. The high temperature behavior of ΔF gives for large t a logarithmic dependence on t, while for lower temperatures we see a crossover from logarithmic behavior at small t to $\Delta F \sim t^{0.2}$ for larger t's. However we have not seen a sharp obvious temperature where this happens. In Fig. 3(b) we see a sharp transition from a low temperature region $T < T_P$, where $\lambda(t)$ is almost t independent, to a high temperature region where $\lambda(t)$ is a decreasing function of t. The transition temperature is very close to the point where $\lambda(T,t)$ is maximal as a function of T for all t.

The observed behavior may be explained as follows. The infinite system is characterized by a correlation length ξ. As long as the size t of the system is smaller than ξ, the system was adequately described in terms of the linear theory of deposition proposed by Edwards and Wilkinson [31]. It is easy to show that within the Edwards-Wilkinson theory $(\Delta S)^2$ is propositional, for a finite system, to $\ln t$, as obtained by us for high temperatures. For t longer than ξ, the nonlinearities become important, and $(\Delta S)^2$ should be extensive in t. How is this related to the temperature dependence we find? The correlation length ξ has a strong dependence on temperature, and $\xi(T)$ is propositional to $\exp[(T/T_0)^{\theta}]$ with $\theta = 3$ according to Fisher and Huse [29], and $\theta=2$ according to Kim, Bray, and Moore [18]. In any case the temperature dependence of $\xi(T)$ is so strong that a relatively small increase in temperature may result in an increase of $\xi(T)$ by orders of magnitude, increasing it from values below the minimal t we are using $(t=50)$ to well above the maximal value $(t=1000)$. Indeed, a more careful examination of the data is consistent with the above explanation. In the temperature region $T_P < T < 2T_P$, all the lines but the one corresponding to the smallest t merge. This suggests that the asymptotic value of $\lambda(t)$ has already been reached, so that the asymptotic value is of order 1. Thus above T_P the asymptotic value is still finite. This may explain the former numerical results that claimed a phase transition in $2+1$ dimensions [22–26]. The strong dependence of the correlation length on temperature suggests that increasing the size of the system does not really undergo a phase transition.

It is not difficult to obtain the asymptotic $(t \to \infty)$ high temperature form of $\lambda(t)$. As expected from the Edwards-Wilkinson model, $\lambda(t)$ decreases as $\ln t/t$. This is true as long as $t > \xi(T)$, but as t becomes of the order of $\xi(T)\lambda$ it must tend to a constant independent on t. Therefore, we expect

$$\lambda(T) \propto \frac{\ln \xi(T)}{\xi(T)} = \left(\frac{T}{T_0}\right)^{\theta} \exp\left[-\left(\frac{T}{T_0}\right)^{\theta}\right]. \qquad (3)$$

Thus we see that although λ is not zero, it decreases extremely quickly with temperature.

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[1] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
[2] A.-L. Barabási and H.E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).
[3] B. Derrida and H. Spohn, J. Stat. Phys. **51**, 817 (1988).
[4] C.A. Bolle, V. Aksyuk, F. Pardo, P.L. Gammel, E. Zeldov, E. Bucher, R. Boie, D.J. Bishop, and D.R. Nelson, Nature (London) **399**, 43 (1999).
[5] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[6] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[7] J. Krug P. Meakin, and T. Halpin-Healy, Phys. Rev. A **45**, 638 (1992).
[8] U. Deker and F. Haake, Phys. Rev. A **11**, 2043 (1975).
[9] D.A. Huse, C.L. Henley, and D.S. Fisher, Phys. Rev. Lett. **55**,

2924 (1985).
[10] D.E. Wolf and J. Kertesz, Europhys. Lett. **4**, 651 (1987).
[11] J.M. Kim and J.M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).
[12] E. Perlsman (unpublished).
[13] T. Halpin-Healy, Phys. Rev. Lett. **62**, 442 (1989).
[14] B.M. Forrrest and T.-H. Tang, Phys. Rev. Lett. **64**, 1405 (1990).
[15] J.M. Kim, M.A. Moore, and A.J. Bray, Phys. Rev. A **44**, 2345 (1991).
[16] M. Schwartz and S.F. Edwards, Europhys. Lett. **20**, 301 (1992); Phys. Rev. E **57**, 5730 (1998).
[17] K. Moser, J. Kertész, and D.E. Wolf, Physica A **178**, 217 (1991).
[18] J. Kim, A.J. Bray, and M.A. Moore, Phys. Rev. A **44**, R4782 (1991).
[19] D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. A **16**, 732 (1977).
[20] E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[21] L.-H. Tang, T. Nattermann, and B.M. Forrest, Phys. Rev. Lett. **65**, 2422 (1990).
[22] B. Derrida and O. Golinelli, Phys. Rev. A **41**, 4160 (1990).
[23] H. Guo, B. Grossmann, and M. Grant, Phys. Rev. Lett. **64**, 1262 (1990).
[24] Y.P. Pellegrini and R. Jullien, Phys. Rev. Lett. **64**, 1745 (1990).
[25] H. Yan, D. Kessler, and L.M. Sander, Phys. Rev. Lett. **64**, 926 (1990); **68**, 1249 (1992).
[26] J.G. Amar and F. Family, Phys. Rev. Lett. **64**, 534 (1990); Phys. Rev. A **41**, 3399 (1990).
[27] P. Meakin, Phys. Rep. **235**, 189 (1993).
[28] J. Kertész and T. Vicsek, in *Fractals in Science*, edited by A. Bunde and S. Havlin (Springer-Verlag, Berlin, 1994).
[29] D.S. Fisher and D.A. Huse [Phys. Rev. B **43**, 10 728 (1991)] studied the entropy fluctuations, and proposed that the entropy fluctuations for sufficient low temperatures in the pinned phase of directed polymers are extensive $[(\Delta S)^2 \propto t]$.
[30] In $1+1$ dimensions, we use a lattice of 2000×2000 for calculation. For $2+1$ dimensions, we study systems up to $t=1000$. In order to study such a large system, we note that the fluctuations in the position of the end point of the polymer are given by $\overline{\langle x^2(t)\rangle} \propto t^{2\nu}$, with $\nu \approx 0.6$ in $2+1$ dimensions. This means that it is possible to choose a constant of order 1 such that the probability that $|\mathbf{x}(t')| > ct^{\nu}$, for any $t' < t$, is exponentially small in t. Therefore, by imposing a lateral restriction on the directed polymer $|\mathbf{x}(t)| < ct^{\nu}$, c is determined. When considering a system with $t=1000$ in $2+1$ dimensions, we use a lattice of $1000\times 80\times 80$. For generating the random numbers, we used the program RAN2 given in W.H. Press and S.A. Teukolsky, Comput. Phys. **6**, 522 (1992).
[31] S.F. Edwards and D.R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).